S.-T. Yau College Student Mathematics Contests 2023

Analysis and Differential Equations

Individual

Please solve three out of the following four problems.

1. Suppose ψ is a Schwartz function (i.e, $\sup(|x|^k+1)|\psi^{(l)}(x)|<\infty$, for all $k,l\geq 0$) with $\int_{-\infty}^{\infty}|\psi(x)|^2dx=1$. Then

$$(\int_{-\infty}^{\infty}x^2|\psi(x)|^2dx)(\int_{-\infty}^{\infty}|\psi^{'}(x)|^2dx)\geq\frac{1}{4}$$

and equality holds if and only if $\psi(x)=Ae^{-Bx^2}, B>0$ such that $\int_{-\infty}^{\infty}|\psi(x)|^2dx=1$.

2. Suppose f is holomorphic in a neighbourhood Ω of the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty}a_nz^n$$

represents the power expansion of f in the open unit disc, then

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=z_0.$$

3. Consider the following planar system:

$$\begin{cases}\dot{x}=xy+x^3,\\ \dot{y}=-y-2x^2.\end{cases}$$

Is the equilibrium $(0,0)$ stable or unstable? Justify your answer.

4. Prove any bounded harmonic function $u(x,y)$ defined in the domain $\{y>x^2\}$ with Dirichlet boundary condition $u(x,x^2)=0$ must be 0.